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REMARKS ON A NONLOCAL TRANSPORT

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ABSTRACT. We consider a one dimensional nonlocal transport equation and its natural multi-dimensional analogues. By using a new pointwise inequality for the Hilbert transform, we give a short proof of a nonlinear inequality first proved by Córdoba, Córdoba and Fontelos in 2005. We also prove several new weighted inequalities for the Hilbert transform and various nonlinear versions. Some of these results generalize to a related family of nonlocal models.

1. INTRODUCTION AND MAIN RESULTS

In this work we consider the following nonlinear and nonlocal transport equation

$$\begin{cases} \theta_t + (\mathcal{H}\theta)\theta_x = -\kappa\Lambda^\gamma\theta, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \theta|_{t=0} = \theta_0, \end{cases} \quad (1.1)$$

where $\theta = \theta(t, x)$ is a scalar-valued function defined on $[0, \infty) \times \mathbb{R}$, and \mathcal{H} is the Hilbert transform defined via

$$\mathcal{H}\theta := \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\theta(y)}{x-y} dy.$$

The number $\kappa \geq 0$ is the viscosity coefficient which governs the strength of the linear dissipation. The dissipation term $\Lambda^\gamma\theta = (-\Delta)^{\gamma/2}\theta$ is defined by using the Fourier transform as

$$\widehat{\Lambda^\gamma\theta}(\xi) = |\xi|^\gamma \widehat{\theta}(\xi),$$

where $0 < \gamma \leq 2$. In other words Λ^γ is the operator corresponding to the Fourier symbol $|\xi|^\gamma$. When $0 < \gamma < 2$ and θ has suitable regularity (for example $\theta \in C^{1,1}$), one has the representation

$$\Lambda^\gamma\theta = C_\gamma \text{PV} \int_{-\infty}^{\infty} \frac{\theta(x) - \theta(y)}{|x-y|^{1+\gamma}} dy,$$

where C_γ is a positive constant depending only on γ . It follows that if θ attains its global maximum at $x = x_*$, then

$$(\Lambda^\gamma\theta)(x_*) \geq 0.$$

By using this and the transport nature of the equation, one has for any smooth solution θ to (1.1) the L^∞ -maximum principle:

$$\|\theta(t, \cdot)\|_\infty \leq \|\theta_0\|_\infty, \quad \forall t > 0.$$

For $\kappa > 0$ and regarding L^∞ as the threshold space, the cases $\gamma < 1$, $\gamma = 1$, $\gamma > 1$ are called supercritical, critical and subcritical respectively. When $\kappa = 0$ the

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model (1.1) becomes the inviscid case and it is deeply connected with the usual two-dimensional surface quasi-geostrophic equation (cf. [11] and the references therein for some recent results). Compared with the usual Burgers equation with fractal dissipation, the model (1.1) in some sense represents the simplest case of a nonlinear transport equation with nonlocal velocity and a viscous fractional dissipation. For some other related one dimensional hydrodynamic models having some connection with the 2D quasi-geostrophic equation and the 3D Euler equation, we refer the reader to [1], [2], [6], [10], [16], [17], [19] and the references therein for additional results.

Concerning the model (1.1), in the inviscid case $\kappa = 0$, Córdoba, Córdoba and Fontelos [7] first proved the breakdown of classical solutions to (1.1) for a generic class of smooth initial data. In [3], by using several new weighted nonlinear inequalities, they proved blow-up for positive, compactly supported initial data but that is not necessarily even. In [4], Castro and Córdoba gave an elegant short proof of the blow-up by considering even functions and by examining $\mathcal{H}(\mathcal{H}ff_{xx})(0) \geq 0$. In [7] when $\kappa > 0$, they also obtained the global well posedness in the subcritical case. For the critical case, global well-posedness can be proved by adapting the method of continuity as in [11]. Blow up for the supercritical case $0 \leq \gamma < 1/2$ was established in [14]. Currently the case $\frac{1}{2} \leq \gamma < 1$ is still open.

For the inviscid case the proof of [7] is based on an ingenious inequality:

$$-\int_{\mathbb{R}} \frac{\mathcal{H}f \cdot f_x}{x^{1+\delta}} dx \geq C_\delta \int_{\mathbb{R}} \frac{f(x)^2}{x^{2+\delta}} dx, \quad (1.2)$$

where $-1 < \delta < 1$, $C_\delta > 0$ is a constant depending only on δ , and f is an even bounded smooth (not necessarily decaying) function on \mathbb{R} with $f(0) = 0$. In the blow-up proof the inequality (1.2) is applied to $f(x) = \theta(0) - \theta(x)$ and thus f in general does not decay at the spatial infinity. The proof of (1.2) in [7] uses Mellin transform and complex analysis. A natural question is whether one can give a completely real variable proof of (1.2). In this direction Kiselev (see [12]) showed that for any even bounded C^1 function f with $f(0) = 0$ and $f' \geq 0$ for $x > 0$, the following inequality (see Proposition 26 therein)

$$-\int_0^1 \frac{\mathcal{H}f(x)f'(x)f(x)^{p-1}}{x^\sigma} dx \geq C_0 \int_0^1 \frac{f(x)^{p+1}}{x^{1+\sigma}} dx, \quad (1.3)$$

where $p \geq 1$, $\sigma > 0$ and C_0 is a positive constant depending on p and σ . Later in [18] Silvestre and Vicol gave four elegant proofs for the inviscid case (one should note that the definition of the Hilbert transform \mathcal{H} used in [18] differs from the usual one by a minus sign! See formula (1.2) therein). The purpose of this paper is to revisit the model (1.1) and give several new and elementary proofs which are all real variable based. In Section 2 we first derive a new point-wise inequality (see Proposition 2.2) for the Hilbert transform acting on even and non-increasing (on $(0, \infty)$) functions on \mathbb{R} , and then we show the Córdoba-Córdoba-Fontelos inequality by a simple application of Hardy's inequality. We also present several simplified arguments whose byproduct lead to a simple proof of the Kiselev inequality (1.3) and further improvements (in particular we disprove the Kiselev inequality without the monotonicity constraint). In Section 3 we generalize the argument to dimensions $n \geq 2$ which works for the generalized surface quasi-geostrophic equations considered in [15, 8, 9, 5]. Note that the blow-up proof here covers the full range of the generalized surface quasi-geostrophic model. In Section 4 we give another

proof which works for general functions having even symmetry (note necessarily monotone decaying) for the Hilbert model case. In Section 5 we generalize the argument to more general α -patch type models.

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2. RADIAL DECREASING FOR DIMENSION $n = 1$

We shall use (often without explicit mentioning) the following Hardy's inequality.

Lemma 2.1 (Hardy). *If $1 \leq p < \infty$, $\tilde{r} \geq 0$ and f is a non-negative measurable function on $(0, \infty)$. Then*

$$\int_0^\infty F(x)^p x^{p-\tilde{r}-3} dx \leq \left(\frac{p}{\tilde{r}}\right)^p \int_0^\infty f(t)^p t^{p-\tilde{r}-1} dt,$$

where $F(x) = \int_0^x f(t) dt$.

Proof. See pp. 35 of [20]. Note that the $F(x)$ defined therein has an extra $1/x$ factor. \square

Proposition 2.2 (A lower bound for Hilbert transform). *Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be an even continuously differentiable function which is non-increasing on $[0, \infty)$. Assume $g' \in L^1 \cap L^\infty$. Then for any $0 < x < \infty$,*

$$(\mathcal{H}g)(x) \geq \frac{2}{\pi} \cdot \frac{1}{x} \int_0^x (g(y) - g(x)) dy.$$

Remark 2.3. For f even, continuously differentiable and non-decreasing on $[0, \infty)$ with $f' \in L^1 \cap L^\infty$, we have the inequality

$$-(\mathcal{H}f)(x) \geq \frac{2}{\pi} \cdot \frac{1}{x} \int_0^x (f(x) - f(y)) dy, \quad \forall 0 < x < \infty.$$

Proof. Since g is even and $g' \leq 0$ on $[0, \infty)$, it is not difficult to check that

$$\begin{aligned} (\mathcal{H}g)(x) &= \frac{1}{\pi} \int_0^\infty \log \left| \frac{x-y}{x+y} \right| g'(y) dy \geq \frac{1}{\pi} \int_0^x \log \left| \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \right| g'(y) dy \\ &\geq \frac{2}{\pi} \int_0^x \frac{y}{x} \cdot (-g'(y)) dy, \end{aligned}$$

where in the last inequality we used

$$-\log \left| \frac{1-\epsilon}{1+\epsilon} \right| \geq 2\epsilon \quad \text{for all } 0 \leq \epsilon < 1.$$

Integration by parts then yields the result. \square

Remark. Another more direct proof (under the same assumptions) is as follows. First observe that for each $0 < x < \infty$,

$$\mathcal{H}g(x) = \frac{2x}{\pi} \int_0^\infty \frac{g(y) - g(x)}{x^2 - y^2} dy.$$

Thanks to monotonicity, the integrand $\frac{g(y)-g(x)}{x^2-y^2} \geq 0$ in either the regime $y < x$ or the regime $y > x$. Thus we can restrict the integral to the regime $0 < y < x$, and obtain

$$\begin{aligned} \mathcal{H}g(x) &\geq \frac{2x}{\pi} \int_{0 < y < x} \frac{g(y) - g(x)}{x^2 - y^2} dy \\ &\geq \frac{2}{\pi} \cdot \frac{1}{x} \int_0^x (g(y) - g(x)) dy. \end{aligned}$$

Proposition 2.2 can now be used to establish the following lemma which is essentially Lemma 2.2 found in [7]. The original proof therein relies on Mellin transform and positivity of certain Fourier multipliers. Our new proof below avoids this and is completely real-variable based. For simplicity we shall make the same assumption on the function g as in Proposition 2.2.

Lemma 2.4. *For any $-1 < \delta < 1$,*

$$-\int_0^\infty \frac{g'(x)(\mathcal{H}g)(x)}{x^{1+\delta}} dx \geq C_\delta \int_0^\infty \frac{(g(x) - g(0))^2}{x^{2+\delta}} dx, \quad \text{where } C_\delta = \frac{1}{\pi} \cdot \frac{(1+\delta)^2}{3+\delta}.$$

Proof. By Proposition 2.2, we have

$$\begin{aligned} \text{LHS} &\geq \frac{2}{\pi} \cdot \int_0^\infty \frac{\int_0^x (g(y) - g(x)) dy \cdot (-g'(x))}{x^{2+\delta}} dx \\ &= \frac{2}{\pi} \int_0^\infty \frac{\int_0^x (f(x) - f(y)) dy \cdot f'(x)}{x^{2+\delta}} dx, \end{aligned}$$

where $f(x) = g(0) - g(x)$. Notice that

$$\frac{d}{dx} \left(\int_0^x (f(x) - f(y)) dy \right) = x f'(x).$$

Now using this and successive integration by parts gives

$$\begin{aligned} &\int_0^\infty \frac{\int_0^x (f(x) - f(y)) dy f'(x)}{x^{2+\delta}} dx \\ &= \int_0^\infty (-1) \cdot \frac{d}{dx} \left(\int_0^x (f(x) - f(y)) dy \frac{1}{x^{2+\delta}} \right) f(x) dx \\ &= - \int_0^\infty \frac{f'(x) f(x)}{x^{1+\delta}} dx + (2+\delta) \int_0^\infty \frac{\int_0^x (f(x) - f(y)) dy f(x)}{x^{3+\delta}} dx \\ &= \left(-\frac{1+\delta}{2} + 2+\delta \right) \int_0^\infty \frac{f(x)^2}{x^{2+\delta}} dx - (2+\delta) \int_0^\infty \frac{\int_0^x f(y) dy f(x)}{x^{3+\delta}} dx \\ &= \frac{3+\delta}{2} \int_0^\infty \frac{(F'(x))^2}{x^{2+\delta}} dx - \frac{3+\delta}{2} (2+\delta) \int_0^\infty \frac{F(x)^2}{x^{4+\delta}} dx, \end{aligned}$$

where $F(x) = \int_0^x f(y) dy$. By Hardy's inequality, we have

$$\int_0^\infty \frac{F(x)^2}{x^{4+\delta}} dx \leq \left(\frac{2}{3+\delta} \right)^2 \int_0^\infty \frac{(F'(x))^2}{x^{2+\delta}} dx.$$

The result then follows. \square

Remark 2.5. One can even give a direct (without using Hardy) proof as follows. Write (after using Proposition 2.2)

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty \frac{\int_0^x (f(x) - f(y)) dy \cdot f'(x)}{x^{2+\delta}} \\ &= \frac{2}{\pi} \iint_{x \geq y} \frac{\frac{1}{2} \frac{d}{dx} ((f(x) - f(y))^2)}{x^{2+\delta}} dx dy \\ &= \frac{2+\delta}{\pi} \iint_{x \geq y} \frac{(f(x) - f(y))^2}{x^{3+\delta}} dx dy. \end{aligned}$$

Now using the inequality $(a - b)^2 = a^2 + b^2 - 2ab \geq (1 - \alpha)a^2 + (1 - \frac{1}{\alpha})b^2$ for any $\alpha > 0$, we obtain

$$\iint_{x \geq y} \frac{(f(x) - f(y))^2}{x^{3+\delta}} dx dy \geq (1 + \frac{1}{2+\delta} - (\alpha + \frac{1}{\alpha} \cdot \frac{1}{2+\delta})) \int_0^\infty \frac{f(x)^2}{x^{2+\delta}} dx.$$

Optimizing in α then yields the inequality with a slightly inferior constant

$$C_\delta = \frac{1}{\pi} \cdot (3 + \delta - 2\sqrt{2+\delta}).$$

Remark 2.6. In the preceding remark, it is possible to obtain the sharper bound by using the following argument. Noting that

$$\int_{x \geq y} \frac{f(y)^2}{x^{3+\delta}} dx dy = \frac{1}{2+\delta} \int_0^\infty \frac{f(y)^2}{y^{2+\delta}} dy,$$

it suffices to treat the term

$$\begin{aligned} \iint_{x \geq y} \frac{f(x)f(y)}{x^{3+\delta}} dx dy &= \int_0^\infty \frac{f(x)}{x^{3+\delta}} \left(\int_0^x f(y) dy \right) dx \\ &= \frac{3+\delta}{2} \int_0^\infty \frac{(\int_0^x f(y) dy)^2}{x^{4+\delta}} dx. \end{aligned}$$

By Cauchy-Schwartz

$$\left(\int_0^x f(y) dy \right)^2 \leq \int_0^x f(y)^2 y^{-p} dy \cdot \frac{x^{p+1}}{p+1}.$$

Interchanging the integral of dx and dy then gives

$$\begin{aligned} \int_0^\infty \frac{(\int_0^x f(y) dy)^2}{x^{4+\delta}} dx &\leq \int_0^\infty \frac{f(y)^2}{y^p} \cdot \frac{1}{p+1} \left(\int_y^\infty x^{-3-\delta+p} dx \right) dy \\ &= \frac{1}{(p+1)(2+\delta-p)} \int_0^\infty \frac{f(y)^2}{y^{2+\delta}} dy. \end{aligned}$$

Choosing $p = \frac{1+\delta}{2}$ then yields the sharper constant

$$C_\delta = (2+\delta) \cdot (1 + \frac{1}{2+\delta} - 2 \cdot \frac{3+\delta}{2} \cdot (\frac{2}{3+\delta})^2) = \frac{(1+\delta)^2}{3+\delta}.$$

2.1. Proof of the Kiselev inequality. We now sketch a simple proof of the Kiselev inequality (1.3). We emphasize that this inequality is stated for *nondecreasing* even functions on \mathbb{R} . For illustration purposes we first consider the simple case $p = 1$. By using Proposition 2.2 (see Remark 2.3), we have

$$\begin{aligned} - \int_0^1 \frac{\mathcal{H}f(x)f'(x)}{x^\sigma} dx &\geq \frac{2}{\pi} \int_0^1 \frac{\int_0^x (f(x) - f(y)) dy f'(x)}{x^{1+\sigma}} dx \\ &= \frac{1}{\pi} \iint_{\substack{0 \leq x \leq 1 \\ 0 < y < x}} \frac{\frac{d}{dx}((f(x) - f(y))^2)}{x^{1+\sigma}} dx dy \\ &\geq \frac{1+\sigma}{\pi} \iint_{\substack{0 \leq x \leq 1 \\ 0 < y < x}} \frac{(f(x) - f(y))^2}{x^{2+\sigma}} dx dy, \end{aligned}$$

where in the last step we have integrated by part in the x -variable and dropped the harmless boundary terms. Note that we can also keep the boundary term and derive a sharper inequality as it is nonnegative. Next we proceed similarly as in Remark 2.5 and derive (below we shall take $0 < \alpha < 1$ and specify its value at the very end)

$$\begin{aligned} &\iint_{0 < y < x < 1} \frac{(f(x) - f(y))^2}{x^{2+\sigma}} dx dy \\ &\geq \iint_{0 < y < x < 1} \frac{(1-\alpha)f(x)^2 + (1-\frac{1}{\alpha})f(y)^2}{x^{2+\sigma}} dx dy \\ &= (1-\alpha) \int_0^1 \frac{f(x)^2}{x^{1+\sigma}} dx + (1-\frac{1}{\alpha}) \int_0^1 f(y)^2 \frac{1}{1+\sigma} \cdot (\frac{1}{y^{1+\sigma}} - 1) dy \\ &\geq (1-\alpha) \int_0^1 \frac{f(x)^2}{x^{1+\sigma}} dx + (1-\frac{1}{\alpha}) \int_0^1 f(y)^2 \frac{1}{1+\sigma} \cdot \frac{1}{y^{1+\sigma}} dy \\ &= (1+\frac{1}{1+\sigma}) \int_0^1 \frac{f(x)^2}{x^{1+\sigma}} dx - (\alpha + \frac{1}{\alpha} \cdot \frac{1}{1+\sigma}) \cdot \int_0^1 \frac{f(x)^2}{x^{1+\sigma}} dx. \end{aligned}$$

Choosing $\alpha = (1+\sigma)^{-\frac{1}{2}}$ then yields the result. Note that in the second inequality above, we used the fact that $0 < \alpha < 1$ so that the term -1 in the y -integral can be safely dropped.

Next we sketch the proof for $1 < p < \infty$. We start with

$$- \int_0^1 \frac{\mathcal{H}f(x)f'(x)}{x^\sigma} f(x)^{p-1} dx \geq \frac{2}{\pi} \underbrace{\int_0^1 \frac{\int_0^x (f(x) - f(y)) dy f'(x)}{x^{1+\sigma}} f(x)^{p-1} dx}_{=: H_1}.$$

Note that for any $0 \leq s \leq 1$, we have the inequality

$$1 - s \geq (1 - s)^p.$$

This in turn implies that (note that below $f(y)/f(x) \leq 1$ for $0 < y < x$ since f is nondecreasing!)

$$(f(x) - f(y))f(x)^{p-1} \geq (f(x) - f(y))^p.$$

Thus

$$\begin{aligned} H_1 &\geq \iint_{0 < y < x < 1} \frac{\frac{1}{p+1} \frac{d}{dx} ((f(x) - f(y))^{p+1})}{x^{1+\sigma}} dx dy \\ &\geq \frac{1+\sigma}{p+1} \iint_{0 < y < x < 1} \frac{(f(x) - f(y))^{p+1}}{x^{2+\sigma}} dx dy. \end{aligned}$$

Now note that for any $\beta > 1$, one can find a constant $c_1 > 0$, depending only on p and β , such that

$$(1-s)^{p+1} \geq c_1 \cdot (1-\beta s^{p+1}), \quad \forall 0 \leq s \leq 1.$$

This in turn implies that

$$(f(x) - f(y))^{p+1} \geq c_1 \cdot (f(x)^{p+1} - \beta f(y)^{p+1}).$$

Using this inequality we then obtain

$$H_1 \geq \text{const} \cdot (1 - \frac{\beta}{1+\sigma}) \int_0^1 \frac{f(x)^{p+1}}{x^{1+\sigma}} dx.$$

Hence taking $1 < \beta < 1 + \sigma$ (say $\beta = 1 + \frac{\sigma}{2}$) then finishes the proof for the case $p > 1$.

2.2. Further remarks. We first point it out that, under the assumption of *monotonicity*, the Kiselev inequality (1.3) is stronger than the Córdoba-Córdoba-Fontelos inequality (1.2). Indeed fix any C^1 bounded even f on \mathbb{R} with $f' \geq 0$ on $(0, \infty)$, apply the Kiselev inequality to $f_L(x) = f(\frac{x}{L})$, and we get (after a change of variable)

$$-\int_0^L \frac{\mathcal{H}f f'}{x^\sigma} dx \geq C_0 \int_0^L \frac{f(x)^2}{x^{1+\sigma}} dx.$$

Note that C_0 is independent of the parameter L . Sending L to infinity and using the Lebesgue Monotone Convergence Theorem (note that the integrand $-\mathcal{H}f \cdot f'$ is non-negative!) then yields the Córdoba-Córdoba-Fontelos inequality for the whole regime $\sigma > 0$. One should note that the same argument yields the inequality

$$-\int_0^\infty \frac{\mathcal{H}f(x) f'(x) f(x)^{p-1}}{x^\sigma} dx \geq C_0 \int_0^\infty \frac{f(x)^{p+1}}{x^{1+\sigma}} dx,$$

where $p \geq 1$ and C_0 depends only on p and σ .

Finally we should point it out that in the Kiselev inequality, the assumption of monotonicity cannot be dropped in general. In what follows we shall construct a counterexample which answers a question raised by Kiselev in [12] (see Remark 1 on page 249 therein).

Proposition 2.7. *For any $\sigma > 0$, there exists an even function $f \in C_c^\infty(\mathbb{R})$ such that*

$$-\int_0^1 \frac{\mathcal{H}f(x) f'(x)}{x^\sigma} dx < 0.$$

In particular we cannot have the Kiselev inequality (1.3) for $p = 1$ without the monotonicity assumption.

Remark. Similarly one can do the case $1 < p < \infty$, but we do not present the details here.

Lemma 2.8. *For any $\sigma > 0$, one can find $\phi_A \in C_c^\infty(\mathbb{R})$, even and supported in $\{0 < |x| < 1\}$, $\phi_B \in C_c^\infty(\mathbb{R})$, even and supported in $|x| > 1$, such that*

$$\int_0^\infty \frac{\mathcal{H}\phi_B(x) \cdot (\phi_A)'(x)}{x^\sigma} dx > 0.$$

Proof of Lemma 2.8. Clearly

$$\int_0^\infty \frac{\mathcal{H}\phi_B(x) \cdot (\phi_A)'(x)}{x^\sigma} dx = - \int_0^\infty \partial_x \left(\frac{\mathcal{H}\phi_B}{x^\sigma} \right) \phi_A(x) dx.$$

Now

$$\partial_x \left(\frac{\mathcal{H}\phi_B}{x^\sigma} \right) = \frac{1}{x^\sigma} (\Lambda\phi_B - \sigma \cdot \frac{1}{x} \cdot \mathcal{H}\phi_B).$$

Observe that (here we use ϕ_B is supported in $|x| > 1$)

$$\begin{aligned} & (\Lambda\phi_B)(1) - \sigma(\mathcal{H}\phi_B)(1) \\ &= \alpha_1 \int_{|y|>1} \frac{-\phi_B(y)}{(1-y)^2} dy - \frac{2}{\pi} \sigma \int_{y>1} \frac{\phi_B(y)}{1-y^2} dy \\ &= - \int_{y>1} \left[\frac{\alpha_1}{(1-y)^2} + \frac{\alpha_1}{(1+y)^2} + \frac{2}{\pi} \sigma \frac{1}{1-y^2} \right] \phi_B(y) dy, \end{aligned}$$

where $\alpha_1 > 0$ is an absolute constant which appear in the definition of the nonlocal operator Λ . It is then clear that one can choose suitable ϕ_B such that

$$(\Lambda\phi_B)(1) - \sigma(\mathcal{H}\phi_B)(1) < 0.$$

By continuity we can find $x_0 < 1$ sufficiently close to 1, such that

$$(\Lambda\phi_B)(x_0) - \sigma \cdot \frac{1}{x_0} (\mathcal{H}\phi_B)(x_0) < 0.$$

Choosing ϕ_A to be a suitable bump function localized around x_0 then yields the result. \square

With the help of Lemma 2.8, we now complete the proof of Proposition 2.7. Choose

$$f(x) = \phi_A(x) + t\phi_B(x).$$

Then clearly (note that below we use the fact that ϕ_B is supported in $|x| > 1$)

$$\begin{aligned} - \int_0^1 \frac{\mathcal{H}f(x)f'(x)}{x^\sigma} dx &= c_1 - t \int_0^1 \frac{\mathcal{H}\phi_B(x) \cdot (\phi_A)'(x)}{x^\sigma} dx \\ &= c_1 - t \int_0^\infty \frac{\mathcal{H}\phi_B(x) \cdot (\phi_A)'(x)}{x^\sigma} dx, \end{aligned}$$

where c_1 is independent of t . Choosing t sufficiently large then yields the result.

3. RADIAL DECREASING FOR DIMENSION $n \geq 2$

In [15, 8, 9] a family of the generalized surface quasi-geostrophic equations were introduced and studied. The simplest inviscid case takes the form:

$$\partial_t g + (\Lambda^{-\alpha} \nabla g \cdot \nabla g) = 0,$$

where $n \geq 2$, $0 < \alpha < 2$ and $\Lambda^{-\alpha}$ corresponds to the Fourier multiplier $|\xi|^{-\alpha}$. These models can be viewed as natural generalizations of the one dimensional Hilbert-type

models to higher dimensions. In what follows we shall discuss the corresponding nonlinear inequalities in analogy with the Hilbert transform case.

Proposition 3.1. *Let $n \geq 2$ and $0 < \alpha < 2$. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radial and non-increasing Schwartz function. Then for any $x \neq 0$,*

$$-(\Lambda^{-\alpha} \nabla g)(x) \cdot \frac{x}{|x|} \geq C_{\alpha,n} \cdot \frac{1}{r^{n-\alpha+1}} \int_0^r (-g'(\rho)) \cdot \rho^n d\rho,$$

where $r = |x|$ and $C_{\alpha,n} > 0$ depends only on (α, n) .

Remark 3.2. Note that for $f(x) = g(0) - g(x)$ radial and nondecreasing, we have

$$(\Lambda^{-\alpha} \nabla f)(x) \cdot \frac{x}{|x|} \geq C_{\alpha,n} \cdot \frac{1}{r^{n-\alpha+1}} \int_0^r (f'(\rho)) \cdot \rho^n d\rho.$$

Proof. Since g is radial we can assume WLOG that $x = re_n = r \cdot (0, \dots, 0, 1)$. By using the fact that $g'(\rho) \leq 0$, we have

$$\begin{aligned} -(\Lambda^{-\alpha} \partial_n g)(x) &= C_{\alpha,n} \int_0^\infty \int_{|\omega|=1} \frac{\omega_n}{|re_n - \rho\omega|^{n-\alpha}} \cdot (-g'(\rho)) \rho^{n-1} d\sigma(\omega) d\rho \\ &\gtrsim \int_0^r (-g'(\rho)) \cdot \rho^{n-1} \cdot r^{-(n-\alpha)} \cdot \frac{\rho}{r} d\rho, \end{aligned}$$

where we have used the simple inequality

$$\int_{|\omega|=1} \frac{\omega_n}{|e_n - \epsilon\omega|^{n-\alpha}} d\sigma(\omega) \gtrsim \epsilon, \quad \text{for } 0 < \epsilon < 1.$$

□

Lemma 3.3. *Let $n \geq 2$, $0 < \alpha < 2$ and $-1 < \delta < 1$. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radial and nonincreasing Schwartz function. Then*

$$\int_{\mathbb{R}^n} \frac{\Lambda^{-\alpha} \nabla g \cdot \nabla g}{|x|^{n+\delta}} dx \geq C_{\alpha,\delta,n} \int_{\mathbb{R}^n} \frac{(g(0) - g(x))^2}{|x|^{n+2-\alpha+\delta}} dx,$$

where $C_{\alpha,\delta,n} > 0$ depends only on (α, δ, n) .

Proof. Denote $f(x) = g(0) - g(x)$. Note that f is non-decreasing and $f(0) = 0$. By Proposition 3.1 and Remark 3.2, we have

$$\begin{aligned} \text{LHS} &\gtrsim \int_0^\infty \frac{\int_0^r f'(\rho) \rho^n d\rho \cdot f'(r)}{r^{n-\alpha+\delta+2}} dr \\ &= - \int_0^\infty \frac{f'(r)f(r)}{r^{2-\alpha+\delta}} dr + (n - \alpha + \delta + 2) \int_0^\infty \frac{\int_0^r f'(\rho) \rho^n d\rho f(r)}{r^{n-\alpha+\delta+3}} dr \\ &= - \frac{2 - \alpha + \delta}{2} \int_0^\infty \frac{f(r)^2}{r^{3-\alpha+\delta}} dr + (n - \alpha + \delta + 2) \int_0^\infty \frac{f(r)^2}{r^{3-\alpha+\delta}} dr \\ &\quad - (n - \alpha + \delta + 2) \int_0^\infty \frac{n \int_0^r f(\rho) \rho^{n-1} d\rho f(r) r^{n-1}}{r^{n-1} \cdot r^{n-\alpha+\delta+3}} dr \\ &= (n + \frac{2 - \alpha + \delta}{2}) \int_0^\infty \frac{(f(r) r^{n-1})^2}{r^{2n+1-\alpha+\delta}} dr \\ &\quad - n(n + 2 - \alpha + \delta) \cdot (n + \frac{2 - \alpha + \delta}{2}) \int_0^\infty \frac{F(r)^2}{r^{2n+3-\alpha+\delta}} dr, \end{aligned}$$

where

$$F(r) = \int_0^r f(\rho) \rho^{n-1} d\rho.$$

Now the result follows from Hardy's inequality (see Lemma 2.1 and take $p = 2$, $\tilde{r} = 2n + 2 - \alpha + \delta$) since

$$1 > n(n + 2 - \alpha + \delta) \cdot \left(\frac{2}{2n + 2 - \alpha + \delta} \right)^2.$$

□

With the help of Lemma 3.3 one can then complete the blow-up proof for the full range of the generalized surface quasi-geostrophic model considered in [15, 8, 9], we omit further details.

4. ANOTHER SHORT PROOF FOR HILBERT

Lemma 4.1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function. Then*

$$-\int_{\mathbb{R}} \frac{g'(x)(Hg)(x)}{x} dx \geq \frac{1}{\pi} \int_{\mathbb{R}} \frac{(g(0) - g(x))^2}{x^2} dx.$$

Proof. By taking advantage of the even symmetry, we have

$$\begin{aligned} \text{LHS} &= \frac{2}{\pi} \int_0^\infty g'(x) \left(\int_0^\infty \frac{g(x) - g(y)}{x^2 - y^2} dy \right) dx \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_0^\infty \left(\int_0^\infty \frac{1}{x^2 - y^2 + i\epsilon} \frac{d}{dx} ((g(x) - g(y))^2) dx \right) dy \\ &= \frac{1}{\pi} \int_0^\infty \frac{(g(0) - g(y))^2}{y^2} dy + \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{(g(x) - g(y))^2}{(x^2 - y^2)^2} \cdot x dx dy \\ &= \frac{1}{\pi} \int_0^\infty \frac{(g(0) - g(y))^2}{y^2} dy + \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{(g(x) - g(y))^2}{(x - y)^2 (x + y)} dx dy. \end{aligned}$$

□

Remark 4.2. The constant $1/\pi$ is certainly not sharp since

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{(g(x) - g(y))^2}{(x - y)^2 (x + y)} dx dy &\geq 2 \int_{x > y} \frac{(g(x) - g(y))^2}{(x^2 - y^2)(x - y)} dx dy \\ &\geq 2 \int_{x > y} \frac{(g(x) - g(y))^2}{x^3} dx dy \\ &\geq (3 - 2\sqrt{2}) \int_0^\infty \frac{(g(0) - g(x))^2}{x^2} dx. \end{aligned}$$

Lemma 4.1 is not directly useful for establishing blow-ups since it involves a non-integrable weight $1/x$. The next lemma fixes this issue.

Lemma 4.3. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function. Then*

$$-\int_{\mathbb{R}} g'(x)(Hg)(x) \frac{e^{-x}}{x} dx \geq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(g(0) - g(x))^2}{x^2} dx - 1000 \|g\|_\infty^2.$$

Proof. By using the same integration by parts argument as in Lemma 4.1, we get

$$\begin{aligned} \text{LHS} &= \frac{1}{\pi} \int_0^\infty \frac{(g(0) - g(y))^2}{y^2} dy + \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{(g(x) - g(y))^2}{(x^2 - y^2)^2} \cdot x e^{-x} dx dy \\ &\quad + \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{(g(x) - g(y))^2}{x^2 - y^2} (e^{-x} - e^{-y}) dx dy. \end{aligned}$$

Now note

$$\int_0^\infty \int_{\frac{x}{2} < y < 2x} \left| \frac{(g(x) - g(y))^2}{x^2 - y^2} \cdot (e^{-x} - e^{-y}) \right| dx dy \leq 100 \|g\|_\infty^2.$$

Also

$$\begin{aligned} &\int_0^\infty \int_{y \leq \frac{x}{2}} \frac{(g(x) - g(y))^2}{|x^2 - y^2|} \cdot |e^{-x} - e^{-y}| dx dy \\ &\leq \frac{8}{3} \int_0^\infty \int_{y \leq \frac{x}{2}} \frac{(g(0) - g(x))^2 + (g(0) - g(y))^2}{x^2} e^{-y} dx dy \\ &\leq \frac{8}{3} \int_0^\infty \frac{(g(0) - g(x))^2}{x^2} (1 - e^{-\frac{x}{2}}) dx + \frac{4}{3} \int_0^\infty \frac{(g(0) - g(y))^2}{y} e^{-y} dy \\ &\leq \frac{1}{2} \int_0^\infty \frac{(g(0) - g(x))^2}{x^2} dx + 300 \|g\|_\infty^2. \end{aligned}$$

The piece $y \geq 2x$ is estimated similarly. \square

To handle the diffusion term, we need the following auxiliary lemma.

Lemma 4.4. *Let $0 < \gamma < 1$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function. Then*

$$\left| \int_0^\infty \frac{\Lambda^\gamma g(0) - \Lambda^\gamma g(x)}{x} e^{-x} dx \right| \leq C_\gamma \int_0^\infty \frac{|g(0) - g(x)|}{x^{1+\gamma}} \log(10 + \frac{1}{x}) dx,$$

where $C_\gamma > 0$ is a constant depending only on γ .

Proof. By using parity, we have

$$(\Lambda^\gamma g)(x) = C_\gamma^{(1)} \int_0^\infty \frac{2g(x) - g(x-y) - g(x+y)}{y^{1+\gamma}} dy,$$

where $C_\gamma^{(1)} > 0$ is a constant depending only on γ . Now

$$\begin{aligned} &\int_0^\infty \frac{\Lambda^\gamma g(0) - \Lambda^\gamma g(x)}{x} e^{-x} dx \\ &= C_\gamma^{(1)} \int_{0 < x < \infty, 0 < y < \infty} \frac{2(g(0) - g(y)) + g(x-y) + g(x+y) - 2g(x)}{xy^{1+\gamma}} e^{-x} dx dy \\ &= C_\gamma^{(1)} \int_{0 < x < \infty, 0 < y < \infty} \underbrace{\frac{-2\tilde{g}(y) + \tilde{g}(x-y) + \tilde{g}(x+y) - 2\tilde{g}(x)}{xy^{1+\gamma}}}_{=: H_1} e^{-x} dx dy, \end{aligned}$$

where for simplicity we have denoted $\tilde{g}(x) := g(x) - g(0)$.

Case 1: $\frac{1}{10} \leq \frac{x}{y} \leq 10$. Clearly

$$\int_{\substack{\frac{1}{10} \leq \frac{x}{y} \leq 10 \\ 0 < x < \infty}} |H_1| dx dy \lesssim \int_0^\infty \frac{|\tilde{g}(x)|}{x^{1+\gamma}} dx.$$

Case 2: $y \geq 10x$. Obviously

$$\int_{\substack{y \geq 10x \\ 0 < x < \infty}} \frac{|\tilde{g}(x)|}{xy^{1+\gamma}} e^{-x} dx dy \lesssim \int_0^\infty \frac{|\tilde{g}(x)|}{x^{1+\gamma}} dx.$$

On the other hand,

$$\begin{aligned} & \left| \int_{\substack{y \geq 10x \\ 0 < x < \infty}} \frac{2\tilde{g}(y) - \tilde{g}(y-x) - \tilde{g}(y+x)}{xy^{1+\gamma}} e^{-x} dx dy \right| \\ &= \left| \int_{0 < x, y < \infty} \frac{\tilde{g}(y)e^{-x}}{x} \cdot \left(\frac{2}{y^{1+\gamma}} \cdot 1_{y \geq 10x} - \frac{1}{(y+x)^{1+\gamma}} \cdot 1_{y \geq 9x} - \frac{1}{(y-x)^{1+\gamma}} \cdot 1_{y \geq 11x} \right) dx dy \right| \\ &\lesssim \int_0^\infty \frac{|\tilde{g}(y)|}{y^{1+\gamma}} dy. \end{aligned}$$

Case 3: $x \geq 10y$. First

$$\int_{\substack{x \geq 10y \\ 0 < y < \infty}} \frac{|\tilde{g}(y)|}{xy^{1+\gamma}} e^{-x} dx dy \lesssim \int_0^\infty \frac{|\tilde{g}(y)|}{y^{1+\gamma}} \cdot \log(10 + \frac{1}{y}) dy.$$

On the other hand,

$$\begin{aligned} & \left| \int_{\substack{x \geq 10y \\ 0 < y < \infty}} \frac{2\tilde{g}(x) - \tilde{g}(x+y) - \tilde{g}(x-y)}{xy^{1+\gamma}} e^{-x} dx dy \right| \\ &= \left| \int_{0 < x, y < \infty} \frac{\tilde{g}(x)}{y^{1+\gamma}} \cdot \left(\frac{2}{x} e^{-x} \cdot 1_{x \geq 10y} - \frac{e^{-(x-y)}}{x-y} \cdot 1_{x \geq 11y} - \frac{e^{-(x+y)}}{x+y} \cdot 1_{x \geq 9y} \right) dx dy \right| \\ &\lesssim \int_0^\infty \frac{|\tilde{g}(x)|}{x^{1+\gamma}} dx. \end{aligned}$$

□

Lemma 4.3 and 4.4 can be used to establish blow up. Consider

$$\begin{cases} \partial_t \theta - H\theta\theta_x = -\Lambda^\gamma \theta, \\ \theta(0, x) = \theta_0(x). \end{cases}$$

Theorem 4.5. *Let $0 < \gamma < \frac{1}{2}$. Let the initial data θ_0 be an even Schwartz function. There exists a constant $A_\gamma > 0$ depending only on γ such that if*

$$\int_0^\infty \frac{\theta_0(0) - \theta_0(x)}{x} e^{-x} dx \geq A_\gamma \cdot (\|\theta_0\|_\infty + 1),$$

then the corresponding solution blows up in finite time.

Proof. By using Lemma 4.3 and 4.4, we compute

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty \frac{\theta(t, 0) - \theta(t, x)}{x} e^{-x} dx \\ &\geq \frac{1}{2\pi} \int_0^\infty \frac{(\theta(t, 0) - \theta(t, x))^2}{x^2} dx - 1000 \|\theta\|_\infty^2 \\ &\quad - C_\gamma \int_0^\infty \frac{|\theta(t, 0) - \theta(t, x)|}{x^{1+\gamma}} \log(10 + \frac{1}{x}) dx. \end{aligned}$$

By Cauchy-Schwartz, it is clear that

$$2 \left(\int_0^\infty \frac{\theta(t, 0) - \theta(t, x)}{x} e^{-x} dx \right)^2 \leq \int_0^\infty \frac{(\theta(t, 0) - \theta(t, x))^2}{x^2} dx.$$

Also by using Cauchy-Schwartz, we have

$$\begin{aligned} & \int_0^1 \frac{|\theta(t, 0) - \theta(t, x)|}{x^{1+\gamma}} \log(10 + \frac{1}{x}) dx \\ & \leq \left(\int_0^1 \frac{(\theta(t, 0) - \theta(t, x))^2}{x^2} e^{-x} dx \right)^{\frac{1}{2}} \cdot \left(\int_0^1 \frac{e^x}{x^{2\gamma}} (\log(10 + \frac{1}{x}))^2 dx \right)^{\frac{1}{2}} \\ & \leq C_1 \cdot \left(\int_0^1 \frac{(\theta(t, 0) - \theta(t, x))^2}{x^2} e^{-x} dx \right)^{\frac{1}{2}}, \end{aligned}$$

where $C_1 > 0$ depends only on γ . Note that here we used the crucial assumption $0 < \gamma < \frac{1}{2}$ for the integral to converge. It then follows easily that

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty \frac{\theta(t, 0) - \theta(t, x)}{x} e^{-x} dx \\ & \geq \frac{1}{2\pi} \left(\int_0^\infty \frac{\theta(t, 0) - \theta(t, x)}{x} e^{-x} dx \right)^2 - C_2 \cdot (\|\theta_0\|_\infty + 1)^2, \end{aligned}$$

where $C_2 > 0$ depends only on γ . Choosing $A_\gamma = \sqrt{2\pi C_2}$ then yields the result. \square

5. THE α -CASE

Remarkably the computation in section 4 can also be generalized to the case with drift term $\Lambda^{-\alpha} \partial_x \theta$. We shall employ the same weight $1/x$.

Lemma 5.1. *Let $0 < \alpha < 1$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function. Then*

$$\int_0^\infty \frac{\Lambda^{-\alpha} g'(x) \cdot g'(x)}{x} dx \geq C_\alpha \cdot \int_0^\infty \frac{(g(0) - g(x))^2}{x^{3-\alpha}} dx,$$

where $C_\alpha > 0$ depends only on α . Similarly for $1 \leq \alpha < 2$, by writing $\Lambda^{-\alpha} \partial_x = -\Lambda^{-(\alpha-1)} \mathcal{H}$, we have

$$-\int_0^\infty \frac{\Lambda^{-(\alpha-1)} \mathcal{H}g(x) \cdot g'(x)}{x} dx \geq C_\alpha \cdot \int_0^\infty \frac{(g(0) - g(x))^2}{x^{3-\alpha}} dx.$$

Remark 5.2. The case $\alpha = 1$ corresponds to $\Lambda^{-1} \partial_x = -\mathcal{H}$ which is the Hilbert transform case which we have treated before.

Proof. We first discuss the case $0 < \alpha < 1$. By using parity, we have

$$\begin{aligned} (\Lambda^{-\alpha} g')(x) &= C_\alpha \int_0^\infty \left(\frac{1}{|x-y|^{1-\alpha}} - \frac{1}{|x+y|^{1-\alpha}} \right) \cdot \frac{d}{dy} (g(y) - g(x)) dy \\ &= C_\alpha \cdot (-1) \cdot \int_0^\infty \frac{d}{dy} \left(\frac{1}{|x-y|^{1-\alpha}} - \frac{1}{(x+y)^{1-\alpha}} \right) (g(y) - g(x)) dy \\ &= C_\alpha \int_0^\infty h(x, y) (g(y) - g(x)) dy, \end{aligned}$$

where

$$h(x, y) = \frac{d}{dx} \left(\frac{1}{|x-y|^{1-\alpha}} + \frac{1}{(x+y)^{1-\alpha}} \right).$$

Now we write

$$\begin{aligned}
& 2 \int_0^\infty \int_0^\infty \frac{h(x, y)(g(y) - g(x))g'(x)}{x} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{h(x, y)}{x} \cdot \frac{d}{dx}((g(x) - g(y))^2) dx dy \\
&= - \int_0^\infty \left(\frac{h(x, y)}{x} (g(x) - g(y))^2 \Big|_{x=0}^\infty \right) dy + \int_0^\infty \int_0^\infty \frac{d}{dx} \left(\frac{h(x, y)}{x} \right) \cdot (g(x) - g(y))^2 dx dy.
\end{aligned}$$

It is easy to check that for some positive constant $C_1 > 0$ (below $y > 0$),

$$(\partial_x h)(0, y) = \frac{d^2}{dx^2} \left(\frac{1}{|x - y|^{1-\alpha}} + \frac{1}{(x + y)^{1-\alpha}} \right) \Big|_{x=0} = C_1 \cdot y^{-(3-\alpha)}.$$

Thus

$$\begin{aligned}
& - \int_0^\infty \left(\frac{h(x, y)}{x} (g(x) - g(y))^2 \Big|_{x=0}^\infty \right) dy \\
&= \int_0^\infty (\partial_x h)(0, y) (g(0) - g(y))^2 dy \gtrsim \int_0^\infty \frac{(g(0) - g(y))^2}{y^{3-\alpha}} dy.
\end{aligned}$$

It remains for us to check that, for all $0 < x, y < \infty$, $x \neq y$,

$$\frac{d}{dx} \left(\frac{h(x, y)}{x} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} \left(\frac{1}{|x - y|^{1-\alpha}} + \frac{1}{(x + y)^{1-\alpha}} \right) \right) \geq 0.$$

By scaling, it suffices prove for all $0 < x < \infty$, $x \neq 1$,

$$\frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} \left(\frac{1}{|x - 1|^{1-\alpha}} + \frac{1}{(x + 1)^{1-\alpha}} \right) \right) \geq 0.$$

We now make a change of variable $x = \sqrt{t}$. Then we only need to prove

$$\frac{d^2}{dt^2} \left(\frac{1}{|\sqrt{t} - 1|^{1-\alpha}} + \frac{1}{(\sqrt{t} + 1)^{1-\alpha}} \right) \geq 0, \quad \forall 0 < t < \infty, t \neq 1.$$

For $1 < t < \infty$, one can get positivity by direct differentiation. For $0 < t < 1$, one can use the fact that the function

$$f(s) = (1 - s)^{-(1-\alpha)} + (1 + s)^{-(1-\alpha)}$$

has a non-negative binomial expansion for $0 < s < 1$.

We now turn to the case $1 \leq \alpha < 2$. The case $\alpha = 1$ is already treated before in Section 4 so we assume $1 < \alpha < 2$. Set $\epsilon = \alpha - 1 \in (0, 1)$. Then it is not difficult to check that

$$-(\Lambda^{-\epsilon} \mathcal{H}g)(x) = C_\alpha \int_0^\infty h(x, y)(g(y) - g(x)) dy,$$

where

$$\begin{aligned}
h(x, y) &= - \left(\frac{|x - y|^\epsilon}{x - y} + \frac{1}{(x + y)^{1-\epsilon}} \right) \\
&= - \frac{1}{\epsilon} \frac{d}{dx} (|x - y|^\epsilon + (x + y)^\epsilon).
\end{aligned}$$

Clearly

$$\partial_x h(0, y) = \text{const} \cdot y^{-(2-\epsilon)}.$$

It then suffices to check for all $0 < t < \infty$, $t \neq 1$,

$$-\frac{d^2}{dt^2}(|\sqrt{t}-1|^\epsilon + |\sqrt{t}+1|^\epsilon) \geq 0.$$

Again for $t > 1$ the inequality follows easily from direct differentiation. For $0 < t < 1$, one just observe that for $0 < s < 1$, the binomial coefficients in the expansion of

$$f(s) = (1+s)^\epsilon + (1-s)^\epsilon = C_0 + \sum_{k \geq 0} C_k s^{2k}$$

satisfies $C_k < 0$ for all $k \geq 1$. \square

Lemma 5.3. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function. If $0 < \alpha < 1$. Then*

$$\int_0^\infty \frac{\Lambda^{-\alpha} g'(x) \cdot g'(x)}{x} e^{-x} dx \geq C_\alpha^{(1)} \cdot \int_0^\infty \frac{(g(0) - g(x))^2}{x^{3-\alpha}} dx - C_\alpha^{(2)} \|g\|_\infty^2,$$

where $C_\alpha^{(1)} > 0$, $C_\alpha^{(2)} > 0$ are constants depending only on α . Similarly for $1 \leq \alpha < 2$, by writing $\Lambda^{-\alpha} \partial_x = -\Lambda^{-(\alpha-1)} \mathcal{H}$, we have

$$-\int_0^\infty \frac{\Lambda^{-(\alpha-1)} \mathcal{H} g(x) \cdot g'(x)}{x} e^{-x} dx \geq C_\alpha^{(3)} \cdot \int_0^\infty \frac{(g(0) - g(x))^2}{x^{3-\alpha}} dx - C_\alpha^{(4)} \|g\|_\infty^2,$$

where $C_\alpha^{(3)} > 0$, $C_\alpha^{(4)} > 0$ depend only on α .

Proof. We only need to modify the proof of Lemma 5.1. Consider first the case $0 < \alpha < 1$. Recall that for $x, y > 0$, $x \neq y$,

$$\begin{aligned} h(x, y) &= \frac{d}{dx} \left(\frac{1}{|x-y|^{1-\alpha}} + \frac{1}{(x+y)^{1-\alpha}} \right) \\ &= -(1-\alpha) \cdot \left(|x-y|^{-(2-\alpha)} \operatorname{sgn}(x-y) + (x+y)^{-(2-\alpha)} \right). \end{aligned}$$

It is not difficult to check that (below $c_i > 0$ are positive constants):

$$\begin{aligned} & c_1 \int_0^\infty \Lambda^{-\alpha} g'(x) \cdot g'(x) \cdot \frac{1}{x} e^{-x} dx \\ &= - \int_0^\infty \int_0^\infty \frac{h(x, y)}{x} e^{-x} \frac{d}{dx} ((g(x) - g(y))^2) dx dy \\ &= c_2 \int_0^\infty \frac{(g(0) - g(y))^2}{y^{3-\alpha}} dy + \int_0^\infty \int_0^\infty \frac{d}{dx} \left(\frac{1}{x} h(x, y) \right) e^{-x} (g(x) - g(y))^2 dx dy \\ &\quad - \int_0^\infty \int_0^\infty h(x, y) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy. \end{aligned}$$

By the computation in Lemma 5.1, we have $\frac{d}{dx} \left(\frac{1}{x} h(x, y) \right) \geq 0$ for any $x, y > 0$, $x \neq y$. Thus we only need to estimate the third term above. Observe that for

$x > y$, we have $h(x, y) < 0$. Then

$$\begin{aligned}
& -\frac{1}{1-\alpha} \int_0^\infty \int_0^\infty h(x, y) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&= \int_0^\infty \int_0^\infty \left(\frac{\operatorname{sgn}(x-y)}{|x-y|^{2-\alpha}} + \frac{1}{(x+y)^{2-\alpha}} \right) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&\geq \iint_{\frac{x}{2} \leq y \leq 2x} \left(\frac{\operatorname{sgn}(x-y)}{|x-y|^{2-\alpha}} + \frac{1}{(x+y)^{2-\alpha}} \right) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&\quad + \iint_{x < \frac{1}{2}y} \left(\frac{\operatorname{sgn}(x-y)}{|x-y|^{2-\alpha}} + \frac{1}{(x+y)^{2-\alpha}} \right) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&\geq \iint_{\frac{x}{2} \leq y \leq 2x} \frac{\operatorname{sgn}(x-y)}{|x-y|^{2-\alpha}} \cdot \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&\quad + \iint_{x < \frac{1}{2}y} \left(\frac{\operatorname{sgn}(x-y)}{|x-y|^{2-\alpha}} + \frac{1}{(x+y)^{2-\alpha}} \right) \frac{1}{x} e^{-x} (g(x) - g(y))^2 dx dy \\
&\geq -\frac{1}{2} \iint_{\frac{x}{2} \leq y \leq 2x} \frac{1}{|x-y|^{2-\alpha}} \left| \frac{1}{x} e^{-x} - \frac{1}{y} e^{-y} \right| (g(x) - g(y))^2 dx dy \\
&\quad - d_1 \iint_{x < \frac{1}{2}y} \frac{1}{y^{3-\alpha}} e^{-x} (g(x) - g(y))^2 dx dy,
\end{aligned}$$

where $d_1 > 0$ is a constant depending only on α . Now observe that for $x, y > 0$ with $x \neq y$ and $\frac{x}{2} \leq y \leq 2x$, we have

$$\frac{1}{|x-y|^{2-\alpha}} \left| \frac{1}{x} e^{-x} - \frac{1}{y} e^{-y} \right| \lesssim \frac{1}{|x-y|^{1-\alpha}} e^{-\frac{1}{10}|x|} (1+x^{-2}).$$

The desired result then follows from the following string of inequalities:

$$\begin{aligned}
& \iint_{\frac{x}{2} \leq y \leq 2x} |x-y|^{-(1-\alpha)} e^{-\frac{1}{10}|x|} (g(x) - g(y))^2 dx dy \lesssim \|g\|_\infty^2; \\
& \iint_{\frac{x}{2} \leq y \leq 2x} |x-y|^{-(1-\alpha)} |x|^{-2} e^{-\frac{1}{10}|x|} (g(x) - g(y))^2 dx dy \lesssim \int_0^\infty \frac{(g(x) - g(0))^2}{x^{2-\alpha}} dx; \\
& \iint_{x < \frac{1}{2}y} \frac{1}{y^{3-\alpha}} e^{-x} (g(x) - g(y))^2 dx dy \lesssim \int_0^\infty \frac{(g(x) - g(0))^2}{x^{2-\alpha}} dx; \\
& \int_0^\infty \frac{(g(x) - g(0))^2}{x^{2-\alpha}} dx \leq \eta \int_0^\infty \frac{(g(x) - g(0))^2}{x^{3-\alpha}} dx + C_{\eta, \alpha} \|g\|_\infty^2,
\end{aligned}$$

where $\eta > 0$ is any small constant, and $C_{\eta, \alpha}$ depends only on (η, α) .

The above concludes the proof for the case $0 < \alpha < 1$. The case for $1 \leq \alpha < 2$ is similar. In that case one only needs to work with $h(x, y)$ given by (up to an unessential positive constant)

$$h(x, y) = \frac{|x-y|^\epsilon}{x-y} + \frac{1}{(x+y)^{1-\epsilon}},$$

where $\epsilon = \alpha - 1 \in [0, 1)$. In the symmetric region $\frac{x}{2} \leq y \leq 2x$, one uses the inequality

$$\left| \frac{|x-y|^\epsilon}{x-y} \cdot \left| \frac{1}{x} e^{-x} - \frac{1}{y} e^{-y} \right| \right| \lesssim |x-y|^\epsilon e^{-\frac{1}{10}x} (1+x^{-2}), \quad \forall x \neq y.$$

In the region $0 < x < \frac{1}{2}y$, one can use the bound

$$\left| \frac{|x-y|^\epsilon}{x-y} + \frac{1}{(x+y)^{1-\epsilon}} \right| \lesssim y^{-(2-\epsilon)} \cdot x.$$

We omit further details. \square

Lemma 5.3 can be used to establish blow up. For simplicity, consider for $0 < \alpha < 1$, the model

$$\begin{cases} \partial_t \theta + (\Lambda^{-\alpha} \partial_x \theta) \cdot \partial_x \theta = 0, \\ \theta(0, x) = \theta_0(x); \end{cases}$$

and for $1 \leq \alpha < 2$, the model

$$\begin{cases} \partial_t \theta - (\Lambda^{-(\alpha-1)} \mathcal{H} \theta) \cdot \partial_x \theta = 0, \\ \theta(0, x) = \theta_0(x); \end{cases}$$

One should check that in both cases, the symbol of the operator for the drift term is given by $i|\xi|^{-\alpha}\xi$ for all $0 < \alpha < 2$. Alternatively, one may write both models as a single equation

$$\partial_t \theta - \Lambda^s \mathcal{H} \theta \cdot \partial_x \theta = 0,$$

where $-1 < s < 1$. The drift term has the symbol $i|\xi|^s \text{sgn}(\xi)$ so that s can be identified as $1 - \alpha$.

Concerning both models, we have the following result.

Theorem 5.4. *Let $0 < \alpha < 2$. Let the initial data θ_0 be an even Schwartz function. There exists a constant $A_\alpha > 0$ depending only on α such that if*

$$\int_0^\infty \frac{\theta_0(0) - \theta_0(x)}{x} e^{-x} dx \geq A_\alpha \cdot (\|\theta_0\|_\infty + 1),$$

then the corresponding solution blows up in finite time.

Remark. One can also consider the model with suitable dissipation term on the right hand side. For simplicity we do not state such results here which can be obtained by using similar estimates as in the previous section.

Proof. This follows from Lemma 5.3. One only needs to use the simple inequality (with respect to the measure $e^{-x} dx$ on $(0, \infty)$) which holds for any $0 < \alpha < 2$:

$$\int_0^\infty \frac{|g(x)|}{x} e^{-x} dx \leq \left(\int_0^\infty \frac{g(x)^2}{x^{3-\alpha}} e^{-x} dx \right)^{\frac{1}{2}} \cdot \left(\int_0^\infty \frac{1}{x^{\alpha-1}} e^{-x} dx \right)^{\frac{1}{2}}.$$

\square

Remark 5.5. Strictly speaking, the proof of Theorem 5.4 assumed the local well-posedness (of smooth solutions) for the generalized model. While the focus of this work is to prove nonlinear Hilbert type inequalities (for showing finite time singularity), for the sake of completeness we sketch the proof of local wellposedness here in this remark. Consider the nontrivial case with hyper-singular velocity as follows:

$$\partial_t \theta - (\Lambda^s \mathcal{H} \theta) \partial_x \theta = 0,$$

where $0 < s < 1$ (the case $-1 < s \leq 0$ is easier). First we present formal energy estimates. For the basic L^2 estimate, we have

$$\frac{1}{2} \frac{d}{dt} (\|\theta\|_2^2) \leq \frac{1}{2} \|\Lambda^{s+1} \theta\|_\infty \|\theta\|_2^2.$$

Next take an integer $m > s + \frac{3}{2}$, and compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\partial_x^m \theta\|_2^2) &= \int \partial_x^m (\Lambda^s \mathcal{H} \theta \partial_x \theta) \partial_x^m \theta dx \\ &= \int (\partial_x^m \Lambda^s \mathcal{H} \theta) \partial_x \theta \partial_x^m \theta dx \end{aligned} \quad (5.1)$$

$$\begin{aligned} &+ \int \Lambda^s \mathcal{H} \theta \partial_x^{m+1} \theta \partial_x^m \theta dx \\ &+ \text{other terms.} \end{aligned} \quad (5.2)$$

It is not difficult to check that (one may take $m \geq 3$ for simplicity, but this can be sharpened)

$$|\text{other terms}| \lesssim \|\theta\|_{H^m}^3.$$

For (5.2) one can do integration by parts and obtain

$$|(5.2)| \lesssim \|\Lambda^{s+1} \theta\|_\infty \|\partial_x^m \theta\|_2^2 \lesssim \|\theta\|_{H^m}^3.$$

To handle (5.1), we can use Lemma 5.6 which gives

$$\|\Lambda^s \mathcal{H}(\partial_x^m \theta \partial_x \theta) - (\Lambda^s \mathcal{H} \partial_x^m \theta) \partial_x \theta\|_2 \lesssim \|\partial_x^m \theta\|_2 \|\partial_x (1 - \partial_{xx})^{\frac{3}{4}} \theta\|_2 \lesssim \|\theta\|_{H^m}^2.$$

Thanks to the skew-symmetry of the Hilbert transform operator, we have

$$\int \Lambda^s \mathcal{H}(\partial_x^m \theta \partial_x \theta) \partial_x^m \theta dx = - \int \partial_x^m \theta \partial_x \theta \Lambda^s \mathcal{H} \partial_x^m \theta dx.$$

We can then rewrite the original term as a commutator and obtain

$$|(5.1)| \lesssim \|\theta\|_{H^m}^3.$$

Thus we have completed the formal energy estimate in H^m . We should point it out that by using the theory in [13] one can obtain sharp energy estimate in H^r with $r > s + \frac{3}{2}$. However we shall not dwell on this issue here.

Finally it is worthwhile pointing it out that in order to make the above formal energy estimates rigorous, one needs to work with the regularized system

$$\partial_t \theta - J_\epsilon (\Lambda^s \mathcal{H} J_\epsilon \theta \partial_x J_\epsilon \theta) = 0,$$

where J_ϵ is the usual mollifier. We leave the interested reader to check the details.

Lemma 5.6. *Let $0 < s < 1$. For any $f, g \in \mathcal{S}(\mathbb{R})$, we have*

$$\|\Lambda^s \mathcal{H}(fg) - (\Lambda^s \mathcal{H}f)g\|_2 \lesssim \|f\|_2 \|\xi|^s \hat{g}(\xi)\|_{L_\xi^1} \lesssim \|f\|_2 \|(1 - \partial_{xx})^{\frac{3}{4}} g\|_2.$$

Remark 5.7. Of course much better results are available. For example by using the commutator estimate in [13] (see Corollary 1.4 and the second remark on page 26 therein), one can even show for any $1 < p < \infty$,

$$\|\Lambda^s \mathcal{H}(fg) - (\Lambda^s \mathcal{H}f)g\|_p \lesssim \|f\|_p \|\Lambda^s g\|_{\text{BMO}}.$$

However for simplicity of presentation (and for the sake of completeness), we present the non-sharp version here.

Proof of Lemma 5.6. Since we are in L^2 it is convenient to work purely on the Fourier side. One can write

$$\mathcal{F}(\Lambda^s \mathcal{H}(fg) - (\Lambda^s \mathcal{H}f)g)(\xi) = -i \cdot \frac{1}{2\pi} \int (|\xi|^s(\operatorname{sgn}(\xi)) - |\eta|^s(\operatorname{sgn}(\eta))) \widehat{f}(\eta) \widehat{g}(\xi - \eta) d\eta.$$

It is easy to check that (since $0 < s < 1$)

$$||\xi|^s(\operatorname{sgn}(\xi)) - |\eta|^s(\operatorname{sgn}(\eta))| \lesssim |\xi - \eta|^s.$$

The result then easily follows from Young's inequality. \square

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